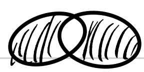
Metric Spaces and Topology
Lecture 15

Baire Measurability. Recall the notion of an ideal of subsets of a given set X: $\mathcal{I} \subseteq \mathcal{P}(X)$ s.t. it's closed downward under $\subseteq$ and closed "upward" under finite unions, e.g. the ideal of finite sets, the ideal of nowhere dense sets.
We also defined σ-ideal as an ideal closed under ctbl unions, e.g. the σ-ideal of ctbl sets, the σ-ideal of meagre sets.

Def. For a set X and an ideal $\mathcal{I}$, we define an eq. rel. $=_{\mathcal{I}}$ on $\mathcal{P}(X)$ as follows: for $A, B \subseteq X$,

$$A =_{\mathcal{I}} B \quad :\iff \quad A \triangle B := (A \setminus B) \cup (B \setminus A) \in \mathcal{I}$$



Claim. $=_{\mathcal{I}}$ is an eq. rel. (is transitive).

Proof. Let $A =_{\mathcal{I}} B =_{\mathcal{I}} C$, and show $A =_{\mathcal{I}} C$.

$$A \triangle C = (A \triangle B) \triangle (B \triangle C) \subseteq (A \triangle B) \cup (B \triangle C) \in \mathcal{I}$$

realize that $(\mathcal{P}(X), \triangle, \emptyset)$ is an abelian group isomorphic to $(2^X, \oplus_2, 0)$. $\square$

Let X be a metric space (more generally, a top. space).
For the σ-ideal $\mathcal{I}$ of meager sets, instead of writing $A =_{\mathcal{I}} B$, we write $A =^* B$.

Def. A subset $B \subseteq X$ is called **Baire measurable** (BM) if $B =^*$ an open set.

Prop. Baire meas. sets form a **σ-algebra**, i.e. are closed under complements and ctbl unions (hence also ctbl intersections and subtraction).

Proof. For complements, if $A =^* U$, U open, then $U =^* \overline{U}$ because $\overline{U} = U \cup \partial U$ and ∂U is n.d.
Thus, $A =^* \overline{U}$ so $A^c =^* (\overline{U})^c$ which is open.
For ctbl unions, let $A_n =^* U_n$, so $\bigcup_n A_n =^* \bigcup_n U_n$ because $\left(\bigcup_n A_n\right) \Delta \left(\bigcup_n U_n\right) \subseteq \bigcup_n (A_n \Delta U_n)$. $\square$

In particular, the σ-algebra BM of Baire meas. sets contains all Borel sets:

Def. In a metric space (more gen. a top. space), the **Borel**

σ-algebra is the smallest σ-algebra containing all open sets. This is just the intersection of all σ-algebras containing the open sets. The sets in this σ-algebra are called Borel.

Examples.

$$\begin{array}{cccc} & & \Sigma^0_2 & \Sigma^0_3 \\ & & \shortparallel & \shortparallel \\ & \text{open} = \Sigma^0_1 & F_\sigma & F_{\sigma\delta} \\ \text{clopen} \subseteq & & & \\ \text{clopen} \subseteq & \text{closed} = \Pi^0_1 & G_\delta & G_{\delta\sigma} \quad \cdots \\ & & \shortparallel & \shortparallel \\ & & \Pi^0_2 & \Pi^0_3 \end{array}$$

More generally: in metric spaces, we have open $\subseteq F_\sigma$, so:

$$\text{clopen} = \Delta^0_1 \begin{array}{c} \subsetneq \Sigma^0_1 \subsetneq \\ \subsetneq \Pi^0_1 \subsetneq \end{array} \Delta^0_2 \begin{array}{c} \subsetneq \Sigma^0_2 \subsetneq \\ \subsetneq \Pi^0_2 \subsetneq \end{array} \Delta^0_3 \begin{array}{c} \subsetneq \Sigma^0_3 \\ \subsetneq \Pi^0_3 \end{array} \cdots \begin{array}{c} \Sigma^0_{\mathbb{N}} \subsetneq \\ \Pi^0_{\mathbb{N}} \subsetneq \end{array} \underbrace{\cdots}_{\omega_1}$$

$$\Pi^0_\xi := \neg \Sigma^0_\xi := \{A \subseteq X : A^c \in \Sigma^0_\xi\}$$

$$\Sigma^0_\xi := \{A \subseteq X : A = \bigcup_n A_n, \text{ where } A_n \in \Pi^0_{\xi'} \text{ for some } \xi' < \xi\}$$

ω_1 := the smallest uncountable ordinal.

In fact, this hierarchy is strict in Polish spaces.

All these sets above are Baire measurable.

Prop. Let X be a metric (top.) space and $B \subseteq X$. TFAE:

(1) B is BM, i.e. $B =^* $ an open set U.

(2) $B = G \cup M$, where G is G_δ and M is meagre.

(3) $B = F \setminus M$, where F is F_σ and M is meagre.

Proof. (2)$\Rightarrow$(1) and (3)$\Rightarrow$(1) follows from the facts that G_δ and F_σ sets are BM.

For (1)$\Rightarrow$(2) and (1)$\Rightarrow$(3), let $\tilde{M}$ be an "upgraded" F_σ meagre set containing $B \Delta U$, and put $G := U \setminus \tilde{M}$ and $F := U \cup \tilde{M}$. $\square$

Remark. In a 2^{nd} ctbl space, there are $\leq$ continuum many Borel sets, while in say $\mathbb{R}$ or any other unctbl Polish space, there are $2^{\text{continuum}}$ many meagre (and hence BM) sets because every subset of meagre is meagre. Thus, there are many more BM sets than Borel sets.

Localization. Let X be a metric (top.) space and U open. We say that a set $B \subseteq X$ is comeagre in U (or U is 100% B) if $B \cap U$ is comeagre in U, equiv. $U \setminus B$ is meagre. We denote this by $U \Vdash B$.

forces

Localization (100% lemma). For a BM set $B \subseteq X$, if it is nonmeagre, then it is comeagre in some nonempty open set U.

Proof. $B =^* U$ so $U \setminus B$ is meagre, hence B is comeagre in U. U can't be $\emptyset$ because B is nonmeagre. $\square$

Obs. If $\overset{\text{open}}{U} \Vdash B$ and $V \subseteq U$ open, then $V \Vdash B$.

Remark. The last observation is one of the main differences between Lebesgue measurable sets and Baire meas. sets. In case of Lebesgue measure, only a 99% lemma holds, so the last observation fails.

Cor (from equivalences to BM). In a perfect Polish space X, every nonmeagre BM set contains a copy of $2^{\mathbb{N}}$.

Proof. A nonmeagre BM set $B \subseteq X$ is $= G \cup M$, where G is G_δ & M is meagre. By an optional HW exercise, G is itself Polish (with an equiv. metric) and is unctbl (because X is perfect, hence ctbl sets are meagre).

By the Cantor-Bendixson theorem, G contains a copy of $2^{\mathbb{N}}$. $\square$

Cor. AC $\Rightarrow$ there non-BM sets. In fact, the Bernstein set is non-BM.

Proof. The Bernstein set is such that it and its complement are both of size continuum but don't contain a copy of $2^{\mathbb{N}}$. One of them must be nonmeagre, hence non-BM by the previous corollary. $\square$

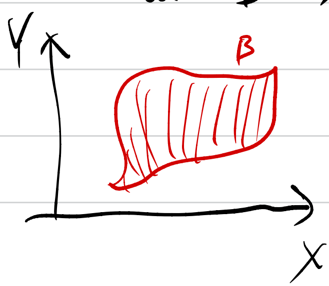
For a set $P \subseteq X$, we write

$$\forall^* x \in X \; \underbrace{x \in P}_{x \text{ satisfies the property } P} \quad :\Longleftrightarrow \quad P \text{ is comeagre.}$$

$$\exists^* x \in X \; x \in P \quad :\Longleftrightarrow \quad P \text{ is nonmeagre.}$$

Observe that $\neg \forall^* x \Longleftrightarrow \exists^* x \, \neg$.

Kuratowski-Ulam Theorem. Let X, Y be 2^{nd} cbl top. spaces. Let $B \subseteq X \times Y$ be a BM subset.



(a) $\forall^* x \in X$ (B_x is BM) and $\forall^* y \in Y$ (B^y is BM).

(b) B is meagre $\Longleftrightarrow \forall^* x \in X$ (B_x is meagre)

$\iff \forall^*_{y \in Y} (B^y \text{ is meagre}).$

Equiv: B is comeagre $\iff$: $\forall^*(x,y) \in X \times Y$ $(x,y) \in B$

$\iff \forall^*_{x \in X} \forall^*_{y \in Y} (x,y) \in B.$

$\iff \forall^*_{y \in Y} \forall^*_{x \in X} (x,y) \in B.$

Cor. Let X be a nonempty perfect Polish space, e.g. $\mathbb{R}$. There is no Baire meas. (as a subset of X^2) well-order $<$ of X. ($<$ is a subset of X^2.)

Def. A well-order $<$ on a set X is a linear (aka total) order on X s.t. every nonempty $Y \subseteq X$ has a $<$-least element. E.g. the usual $<$ on $\mathbb{N}$. The usual $<$ on $\mathbb{R}$ is not a well-order because there is no least positive element.